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Trailing-edge stall

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A study is made of the laminar flow in the neighbourhood of the trailing edge of an aerofoil at incidence. The aerofoil is replaced by a flat plate on the assumption that leading-edge stall has not taken place. It is shown that the critical order of magnitude of the angle of incidence α^* for the occurrence of separation on one side of the plate is $\alpha^* = O(R^{-\frac{1}{16}})$, where R is a representative Reynolds number, for incompressible flow, and $\alpha^* = O(R^{-\frac{1}{4}})$ for supersonic flow. The structure of the flow is determined by the incompressible boundary-layer equations but with unconventional boundary conditions. The complete solution of these fundamental equations requires a numerical investigation of considerable complexity which has not been undertaken. The only solutions available are asymptotic solutions valid at distances from the trailing edge that are large in terms of the scaled variable of order $R^{-\frac{3}{8}}$, and a linearized solution for the boundary layer over the plate which gives the antisymmetric properties of the aerofoil at incidence. The value of α^* for which separation occurs is the trailing-edge stall angle and an estimate is obtained from the asymptotic solutions. The linearized solution yields an estimate for the viscous correction to the circulation determined by the Kutta condition.

1. Introduction

The flow near the trailing edge of a flat plate aligned with a uniform stream in an incompressible viscous fluid has recently been studied by both Stewartson (1969) and Messiter (1969). Both authors showed that when the Reynolds number R is large the flow in the neighbourhood of the trailing edge of the plate has a complicated three-layer or triple-deck structure. This triple deck is similar to that encountered by Stewartson & Williams (1969) in their investigation of the self-induced separation of supersonic flow. In the sublayer, of thickness $O(R^{-\frac{5}{8}})$, the appropriate equations are the incompressible boundary-layer equations but with boundary conditions involving a match with the main deck, which is essentially inviscid; additionally, in the trailing-edge problem, matching is necessary both with the Blasius (1908) solution upstream and the Goldstein (1930) wake solution downstream. The numerical solution of the sublayer equations successfully carried out by Stewartson & Williams (1969) was aided by the fact that the upper-deck equation in the supersonic case is the wave equation rather than the potential equation. This leads to a slightly simplified outer boundary condition in the lower deck.

The cause of this triple deck in the trailing-edge problem is the change in boundary condition at the trailing edge O from zero tangential velocity to zero stress on the line of symmetry. The effect of the triple deck is to induce a favourable pressure gradient upstream of O. The transition of the solution through O is achieved by the Rott & Hakkinen (1965) similarity solution. Downstream of the trailing edge the pressure increases, slightly overshooting its main-stream value before tending to it from above.

The present paper extends the work of Stewartson (1969) to the case when the plate is at a small angle of incidence to the oncoming stream. The purpose of the study is to estimate the circulation around a two-dimensional aerofoil at incidence when the effect of viscosity is taken into account. When the viscosity is zero this is determined by the Kutta condition. We also aim to elucidate some of the phenomena of trailing-edge stall. We make the assumption that the ratio of the thickness of the aerofoil to the angle of incidence is large enough for the fluid not to separate at the leading edge, and that the flow remains attached over the forward part of the body. Thus the boundary layer approaches the trailing edge in an adverse pressure gradient on the upper side of the aerofoil, though the incidence induces a favourable pressure gradient on the lower side. Within a distance $O(R^{-\frac{3}{8}})$ of O the effect of the triple deck, discussed above for the symmetrically disposed plate, makes itself felt. The boundary layer on the upper side of the aerofoil thus experiences a favourable pressure gradient which tends to counteract the adverse gradient due to the incidence. If the angle of incidence is large the flow separates before it is influenced by the triple deck, and if the angle is too small the effect of the triple deck outweighs that of the incidence and the boundary layer remains attached right to the trailing edge. If, however, the angle of incidence α^* is $O(R^{-\frac{1}{16}})$ the two effects are comparable, and we postulate the existence of a critical angle $R^{-\frac{1}{16}}\alpha_s$ at which trailing-edge stall is liable to occur since the flow just separates on the upper side of the aerofoil.

In order to bring out the essential features of the trailing-edge problem unencumbered by complicated geometry, we replace the aerofoil by a flat plate at incidence in a uniform stream. This simplifies the main-stream velocity, and justification for the replacement is discussed in § 2. The flow upstream of the trailing edge is then the Blasius flow plus a perturbation that is $O(\alpha^*)$. At this stage the flow on the lower side of the plate is obtained from that on the upper side by changing the sign of α^*. These two boundary layers then separately enter the triple deck which is centred on O and of thickness $O(R^{-\frac{5}{8}})$. The equations that then hold in the lower deck have more complicated boundary conditions than in the case of the symmetrically disposed plate, as the unknown functions that appear in them are no longer the same on both sides of the plate. At O the boundary condition of zero velocity on the plate is abandoned and instead the pressure must be continuous across the wake. Downstream the solution must finally become that of the Goldstein wake though with the centre line displaced. The equations for the fundamental problem of the lower deck are set up, though a full numerical solution has not yet been undertaken. However, it is shown that the partial differential equations have the correct asymptotic behaviour both upstream, where they match with the perturbed Blasius solution, and downstream where they match with the modified Goldstein wake mentioned above. In this asymmetric problem the transition through the trailing edge itself is achieved by an extension of the Rott & Hakkinen (1965) solution for the symmetrically disposed plate.

Although the complete solution remains to be computed for the plate at both zero and non-zero incidence, the asymmetry of the flow about the aerofoil at incidence enables some of the features of the flow to be deduced from a linearized solution of the equation. Upstream of the trailing edge it is reasonable to linearize about the linear shear with which the streamwise velocity must match at the outer edge of the lower deck. A solution of the resulting equation for the difference in the streamwise velocity components on the top and bottom of the plate is then obtainable by Wiener–Hopf arguments without the need to solve for the boundary layer in the wake. This equation involves the anti-symmetric part of the unknown pressure which must vanish downstream of the trailing edge. The resulting solution, whose asymptotic form is correct both upstream and downstream of O, is consistent with the predicted behaviour of the solution of the full non-linear equations and leads to an estimate of the viscosity correction to the circulation given by the Kutta condition.

The final section of the paper describes the modifications required if the fluid is compressible. If the flow is subsonic the incompressible results carry over with a scaling involving the Mach number and temperature at infinity. If it is supersonic the critical angle of incidence for separation and trailing-edge stall to occur is $\alpha^* = O(R^{-\frac{1}{4}})$. In this case the boundary conditions for the lower-deck problem are similar to those of Stewartson & Williams (1969).

2. The exterior inviscid flow

Consider a two-dimensional aerofoil of length l with a sharp trailing edge in an infinite incompressible fluid of density ρ and kinematic viscosity ν. At infinity the velocity of the fluid is uniform and of magnitude U_∞, and the aerofoil, which is without camber, is fixed at an angle α^* to the direction of the undisturbed stream. The design of the aerofoil is such that the flow over it is smooth and attached except possibly in the immediate neighbourhood of the trailing edge. For leading-edge separation to be avoided it is necessary to have the thickness ratio τ of the body very much greater than the angle of attack. If $\tau = O(\alpha^*)$, so that the two quantities are of the same order, the initial stagnation point is followed by a region of rapid pressure fall on one side of the aerofoil and then by a region of adverse pressure gradient which can provoke separation and long or short bubbles of reversed flow. Here we wish to exclude this phenomenon so that we can concentrate on trailing-edge stall and so we take $\tau \gg \alpha^*$. However, we wish to keep the external inviscid flow as simple as possible and consequently it would be convenient to replace the aerofoil by a flat plate at incidence α^* since this is sufficient to bring out the essential features of the trailing-edge flow. Since trailing-edge stall is estimated to occur when $\alpha^* = O(R^{-\frac{1}{16}})$, consideration of this simpler geometry may formally be justified if we suppose, for example, that the aerofoil has thickness ratio $\tau = O(R^{-\frac{1}{32}})$ in which case leading-edge separation will not occur. Here $R = U_\infty l/\nu$ is the Reynolds number and is taken to be large, though the flow is assumed to remain laminar and steady throughout. The effect of a non-zero trailing-edge angle β^* is of secondary importance if β^* is sufficiently small since, as shown by Riley & Stewartson (1969), the flow does not separate over a symmetrically-disposed wedge if $\beta^* \ll R^{-\frac{1}{4}}$. A similar situation occurs for a cusped trailing edge. We take $\beta^* \ll R^{-\frac{1}{4}}$, so that the effect of the trailing-edge angle is negligible compared with that of the incidence. Although we shall concentrate on a flat plate from now on, our results may easily be generalized to include aerofoils of thickness ratio $\tau = O(1)$. The only modification necessary is to the external inviscid flow, which in the neighbourhood of the trailing edge has the same structure as for a flat plate.

The plate is taken to occupy the strip $-l < x^* < 0$ of the x^* axis with the origin of co-ordinates at the trailing edge. The velocity components in the x^*, y^* directions are u^*, v^* respectively and at an infinite distance upstream, i.e. as $x^* \to -\infty$, we have, making use of the assumption that $\alpha^* \ll 1$,

$$u^* \to U_\infty, \quad v^* \to U_\infty \alpha^*. \tag{2.1}$$

Since leading-edge separation has not occurred, and the Reynolds number is large, it is legitimate to expect that the flow is inviscid almost everywhere, the exceptions being the neighbourhood of the flat plate and the wake extending downstream from the trailing edge. The inviscid solution outside these regions is well-known and has the properties that on the flat plate ($y^* = 0$, $-l < x^* < 0$)

$$v^* = 0, \quad u^* = U_\infty - U_\infty \alpha^* \frac{x^* + B}{[(-x^*)(l + x^*)]^{\frac{1}{2}}} \operatorname{sgn} y^*, \tag{2.2}$$

and on the wake centre line ($y^* = 0$, $x^* > 0$)

$$u^* = U_\infty, \quad v^* = U_\infty \alpha^* \frac{x^* + B}{[x^*(l + x^*)]^{\frac{1}{2}}}, \tag{2.3}$$

where B is a constant to be determined.

The constant B is usually determined by the Kutta condition applied at the trailing edge. An interpretation of this condition, which implies $B = 0$, is that a stagnation point on the upper side of the plate near the trailing edge is to be excluded. If such a stagnation point occurs it is argued that the boundary layer must separate further upstream on that side of the plate. The lift coefficient derived from (2.2) is

$$C_L = 2\pi\alpha^* \left(1 - \frac{2B}{l}\right), \tag{2.4}$$

and if $B = 0$ it is broadly in line with experiment for small α^*. However, the inviscid theory, in conjunction with the Kutta condition, does not explain why, at some value of α^*, usually between 5° and 15°, catastrophic stall, nevertheless, sets in. The contribution to the theoretical explanation of the observed flow properties to be made here may be summarized as follows. First, if α^* is sufficiently small no separation occurs. The reason for this is that the change in character of the boundary layer as the trailing edge is approached induces a favourable pressure gradient which dominates the adverse pressure gradient implied by (2.2). Secondly, when α^* is of a critical order of magnitude, in fact when $\alpha^* = O(R^{-\frac{1}{16}})$, the boundary layer associated with the main stream (2.2) must always separate before $x^* = 0$. Stall begins, therefore, when α^* is large enough to cause the boundary layer to separate before the induced favourable pressure gradient is able to make its impact. In these circumstances it emerges that the constant $B = O(R^{-\frac{3}{8}}l)$ which results in a stagnation point of the inviscid flow at a distance $-x^*/l = O(R^{-\frac{7}{8}})$ upstream of the trailing edge. In order to quantify this argument we now consider the boundary layer corresponding to the main stream (2.2).

3. The perturbed Blasius flow

Apart from the immediate neighbourhood of the leading and trailing edges the velocity of slip implied by (2.2) is virtually uniform. For the reasons outlined in the previous section the singularity in (2.2) at $x^* = -l$, the leading edge of the plate, is ignored, and on the upper side of the plate, to which we shall restrict attention in this section, we replace the main-stream velocity by $U_1(x^*)$ where

$$U_1(x^*) = U_\infty + U_\infty \alpha^* (-x^*/l)^{\frac{1}{2}}. \tag{3.1}$$

Thus we have simplified the slip velocity and set $B = 0$. The first modification leads to an error in both the slip velocity and its derivative that is small over the whole of the plate, except for the leading edge, and therefore will not make a significant contribution to the theory below. The second modification anticipates that $B = O(R^{-\frac{3}{8}}l)$ and may be justified *a posteriori*. We note, however, that even if $B = O(l)$ the main properties of the perturbed Blasius flow can easily be inferred from the discussion when $B = 0$.

We define the parameter e by

$$e^{-8} = R = U_\infty l/\nu, \tag{3.2}$$

and introduce the non-dimensional variables

$$\xi = 1 + x^*/l, \quad \bar{y} = y^*/l\epsilon^4, \quad u = u^*/U_\infty, \quad \bar{v} = v^*/U_\infty \epsilon^4, \tag{3.3}$$

in terms of which the boundary-layer equations appropriate to the main stream (3.1) are

$$\frac{\partial u}{\partial \xi} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad u\frac{\partial u}{\partial \xi} + \bar{v}\frac{\partial u}{\partial \bar{y}} = \frac{-\alpha^*}{2(1-\xi)^{\frac{1}{2}}}[1 + \alpha^*(1-\xi)^{\frac{1}{2}}] + \frac{\partial^2 u}{\partial \bar{y}^2}. \tag{3.4}$$

These equations are to be solved subject to the boundary conditions

$$u = \bar{v} = 0 \quad \text{on} \quad \bar{y} = 0; \quad u \to 1 + \alpha^*(1-\xi)^{\frac{1}{2}} \quad \text{as} \quad \bar{y} \to \infty, \tag{3.5}$$

and if $\alpha^* = 0$ the solution is $u = f_B'(\zeta)$ where $\zeta = \bar{y}/\xi^{\frac{1}{2}}$ and $f_B(\zeta)$ is the Blasius function with $f_B(0) = f_B'(0) = 0$, $f_B''(0) = \lambda = 0{\cdot}3321$. When α^* is small but non-zero we seek a perturbation to the Blasius solution in the manner described by Riley & Stewartson (1969) in their analogous investigation in the case of a wedge. We write

$$u = f_B'(\zeta) + \alpha^* \sum_{n=0}^{\infty} \xi^{n+1} f_n'(\zeta) + O(\alpha^{*2}) \tag{3.6}$$

in (3.3), and the equation satisfied by the function $f_n'(\zeta)$ is

$$f_n''' + \tfrac{1}{2} f_B f_n'' - (n+1) f_B' f_n' + (n + \tfrac{3}{2}) f_B'' f_n = \frac{(n-\frac{1}{2})!}{2\pi^{\frac{1}{2}} n!}, \tag{3.7}$$

with boundary conditions

$$f_n(0) = f_n'(0) = 0, \quad f_n'(\infty) = -\frac{(n-\frac{1}{2})!}{2\pi^{\frac{1}{2}}(n+1)!}. \tag{3.8}$$

Since we are interested in the singular behaviour of the solution (3.6) as $\xi \to 1$, the trailing edge of the plate, we examine the functions f_n for large values of n by writing

$$2\pi^{\frac{1}{2}} n^{\frac{3}{2}} f_n(\zeta) = \Phi_n(\zeta) + o(1). \tag{3.9}$$

The equation for Φ_n is exactly that considered by Riley & Stewartson (1969) and in the same way it follows that

$$\Phi_n'(\zeta) = -\frac{n^{\frac{1}{3}}(-\frac{2}{3})!}{3^{\frac{2}{3}}\lambda^{\frac{5}{3}}} f_B''(\zeta) \tag{3.10}$$

if $\zeta = O(1)$, but if ζ is small so that $n^{\frac{1}{3}}\zeta = O(1)$, then

$$\Phi_n'(\zeta) = -\frac{n^{\frac{1}{3}} 3^{\frac{1}{3}}(-\frac{2}{3})!}{\lambda^{\frac{1}{3}}} \int_0^{n^{\frac{1}{3}}\zeta} \mathrm{Ai}\,(\lambda^{\frac{1}{3}} t)\, dt, \tag{3.11}$$

since (3.10) does not satisfy the boundary condition at the wall. Here Ai is the Airy function.

Thus, near $\xi = 1$,

$$u \approx f_B'(\bar{y}) - \alpha^* \frac{(-\frac{2}{3})!}{3^{\frac{2}{3}} 2\pi^{\frac{1}{2}} \lambda^{\frac{5}{3}}} f_B''(\bar{y}) \sum_{n=1}^{\infty} \frac{\xi^n}{n^{\frac{7}{6}}} \tag{3.12}$$

for any fixed $\bar{y} > 0$, while the skin friction must be calculated from the expression

$$u \approx f_B'(\bar{y}) - \alpha^* \frac{3^{\frac{1}{3}}(-\frac{2}{3})!}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{3}}} \sum_{n=1}^{\infty} \frac{\xi^n}{n^{\frac{7}{6}}} \int_0^{n^{\frac{1}{3}}\bar{y}} \mathrm{Ai}\,(\lambda^{\frac{1}{3}} t)\, dt. \tag{3.13}$$

Since the terms in the series in (3.12), (3.13) were deduced from the properties of (3.7) for large n only, these solutions may be augmented by any term $O(\alpha^*)$ having an expansion in powers of ξ which, when $\xi = 1$, converges more rapidly than the term given. It follows from (3.12) that, near $\xi = 1$,

$$u \approx f_B'(\bar{y}) + \alpha^* \left\{ g_{\frac{1}{2}}(\bar{y}) + \frac{6^{\frac{1}{3}}(-\frac{1}{3})!}{\lambda^{\frac{5}{3}}} (1-\xi)^{\frac{1}{6}} f_B''(\bar{y}) \right\}, \tag{3.14}$$

where the singular part of $g_{\frac{1}{2}}(\bar{y})$ as $\bar{y} \to 0$ is obtained by letting $\xi \to 1$ in the expression for u in (3.13). Thus for small $\bar{y}$ we have

$$g_{\frac{1}{2}}(\bar{y}) \approx -\frac{3^{\frac{1}{3}}(-\frac{2}{3})!}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{3}}} \sum_{n=1}^{\infty} n^{-\frac{5}{6}} \int_0^{\bar{y}} \mathrm{Ai}\,(n^{\frac{1}{3}}\lambda^{\frac{1}{3}} t)\, dt. \tag{3.15}$$

In (3.14) the error in replacing $f_B'(\zeta)$ by $f_B'(\bar{y})$ is $O(1-\xi)$, and the term in α^* is in error by powers of $1-\xi$ higher than $(1-\xi)^{\frac{1}{6}}$.

The behaviour of $g_{\frac{1}{2}}(\bar{y})$ for small $\bar{y}$ is most easily found by consideration of the shear stress which may be deduced from (3.13). It is proportional to $\partial u/\partial \bar{y}$ where

$$\frac{\partial u}{\partial \bar{y}} \approx f_B''(\bar{y}) - \alpha^* \frac{3^{\frac{1}{3}}(-\frac{2}{3})!}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{3}}} \sum_{n=1}^{\infty} \frac{\xi^n}{n^{\frac{5}{6}}} \mathrm{Ai}\,(n^{\frac{1}{3}}\lambda^{\frac{1}{3}}\bar{y}), \tag{3.16}$$

to which may be added any term $O(\alpha^*)$ which when $\bar{y} = 0$ and $\xi = 1$ diverges less strongly than $\sum_{n=1}^{\infty} n^{-\frac{5}{6}}$. Investigation of (3.16) reveals that the double limiting process $\bar{y} \to 0$, $\xi \to 1$ is non-commutative. If we let $\bar{y} \to 0$ first we obtain

$$\left.\frac{\partial u}{\partial \bar{y}}\right|_{\bar{y}=0} \approx \lambda - \alpha^* 6^{-\frac{1}{3}} \lambda^{-\frac{1}{3}} \left[\frac{(-\frac{2}{3})!}{(-\frac{1}{3})!}\right]^2 (1-\xi)^{-\frac{1}{6}}. \tag{3.17}$$

However
$$\frac{\partial u}{\partial \bar{y}}\bigg|_{\xi=1} \approx f_B''(\bar{y}) - \alpha^* \frac{3^{\frac{1}{3}}(-\frac{2}{3})!}{2\pi^{\frac{3}{2}}\lambda^{\frac{2}{3}}} \sum_{n=1}^{\infty} n^{-\frac{5}{6}} \int_0^{\infty} \cos\left(\frac{s^3}{3\lambda} + n^{\frac{1}{3}}\bar{y}s\right) ds, \tag{3.18}$$

where the Airy function has been replaced by an integral representation. In order to investigate the behaviour of (3.18) as $\bar{y} \to 0$ we consider the summation which is the real part of the integral

$$j(\bar{y}) = \int_0^{\infty} \exp(is^3/3\lambda) \sum_{n=1}^{\infty} \frac{\exp(i\bar{y}sn^{\frac{1}{3}})}{n^{\frac{5}{6}}} ds, \tag{3.19}$$

since the interchange of summation and integration may be justified. The series in (3.19) converges for all $\bar{y} > 0$, and because

$$\sum_{n=1}^{\infty} \frac{\exp(in^{\frac{1}{3}}\theta)}{n^{\frac{5}{6}}} = \int_1^{\infty} \frac{\exp(it^{\frac{1}{3}}\theta)}{t^{\frac{5}{6}}} dt + O(1) \quad \text{as} \quad \theta \to 0+ \tag{3.20}$$

it follows that, for small $\bar{y}$,

$$j(\bar{y}) \approx \frac{3\pi^{\frac{1}{2}} \exp(i\pi/4)}{\bar{y}^{\frac{1}{2}}} \int_0^{\infty} \frac{\exp(is^3/3\lambda)}{s^{\frac{1}{2}}} ds. \tag{3.21}$$

Hence finally, for small $\bar{y}$,

$$\frac{\partial u}{\partial \bar{y}}\bigg|_{\xi=1} \approx \lambda - \alpha^* \frac{3^{\frac{1}{3}}(-\frac{2}{3})!\,(-\frac{5}{6})!}{4\pi\lambda^{\frac{1}{2}}} \bar{y}^{-\frac{1}{2}}, \tag{3.22}$$

so that $g_{\frac{1}{2}}(\bar{y})$ differs from
$$-\frac{3^{\frac{1}{3}}(-\frac{2}{3})!}{\lambda^{\frac{1}{2}}(-\frac{1}{6})!} \bar{y}^{\frac{1}{2}} \tag{3.23}$$

by a constant as $\bar{y} \to 0$ and has a singular derivative at $\bar{y} = 0$.

It is at this stage that we first have confirmation of the prediction of § 2 regarding the order of magnitude of α^*. We know (Stewartson 1969, to which we hereafter refer as I) that if $\alpha^* = 0$ the Blasius flow breaks down when $1 - \xi = O(\epsilon^3)$ since the trailing edge induces a favourable pressure gradient. If the adverse pressure gradient caused by the incidence is to be comparable, we see from (3.17) that $\alpha^* = O(\epsilon^{\frac{1}{2}})$. This is also consistent with (3.22) since within a distance $O(\epsilon^3)$. of the trailing edge the appropriate scale for $\bar{y}$ in the immediate neighbourhood of the wall is $\bar{y} = O(\epsilon)$.

The following section describes the modification to the trailing-edge triple deck of I to accommodate the singular behaviour of $\partial u/\partial \bar{y}$ as demonstrated in (3.17), (3.22) in the respective limits $\bar{y} \to 0$ for fixed $\xi \neq 1$ and $\xi \to 1$ for fixed $\bar{y} > 0$. It will emerge, as is indicated by (3.13), that the appropriate combination of co-ordinates in the neighbourhood of the wall is $\bar{y}/(1-\xi)^{\frac{1}{3}}$, a variable that remains $O(1)$ in the scaled co-ordinates of the lower deck.

4. The trailing-edge triple deck for $y^* > 0$

Even if $\alpha^* = 0$ and the Blasius flow is maintained over $-l < x^* < 0$, it has already been shown in I that it must break down within a distance $O(\epsilon^3 l)$ of the trailing edge. Also as $x^* \to 0-$ the work of the previous section already shows that the boundary layer is taking on the familiar properties of the lower and main decks, of thicknesses $O(\epsilon^5 l)$ and $O(\epsilon^4 l)$ respectively. Further, the normal velocity associated with the main deck is seen from (3.14) to be

$$O(U_\infty \alpha^* \epsilon^4 (-x^*/l)^{-\frac{5}{6}}) \tag{4.1}$$

and is of the same order as the term $O(\alpha^*)$ in the slip velocity (3.1) when $-x^* = O(\epsilon^3 l)$. Now, as demonstrated in I, the increase in slip velocity induced by the change in boundary condition at $y^* = 0$ when x^* changes sign is $O(\epsilon^2 U_\infty)$ which is comparable with (4.1) in the triple deck if $\alpha^* = O(\epsilon^{\frac{1}{2}})$. For larger values of α^* separation occurs for $-x^* \gg \epsilon^3 l$ on the upper side of the plate, and for smaller values of α^* the effect of the incidence is negligible in comparison with the trailing-edge effect. Accordingly, interest centres on values of α^* such that $\alpha = O(1)$ where

$$\alpha^* = \epsilon^{\frac{1}{2}} \lambda^{\frac{9}{8}} \alpha, \tag{4.2}$$

where $\lambda = f''_B(0) = 0{\cdot}3321$ and is introduced here merely to simplify the fundamental equation (3.14).

In setting up the triple deck the arguments given in Stewartson & Williams (1969) and in I are used extensively. The main modifications necessary are to the boundary conditions which depend on §§2, 3. Otherwise the structure is taken over, with notation, from I. We write

$$x^* = \epsilon^3 \lambda^{-\frac{5}{4}} l x, \quad y^* = \epsilon^4 \lambda^{-\frac{5}{8}} l y, \quad u^* = U_\infty u, \quad v^* = U_\infty \lambda^{\frac{5}{8}} v,$$
$$p^* = p_\infty + \rho U_\infty^2 \lambda^{\frac{5}{8}} p, \tag{4.3}$$

where u, v, p are functions of x, y. Then in the main deck $x = O(1)$, $y = O(1)$ and we set up the following formal expansions for u, v, p:

$$u(x,y) = U_0(y) + \epsilon^{\frac{1}{2}} u_{\frac{1}{2}}(y) + \epsilon \log \epsilon\, u_{11}(y) + \epsilon u_1(x,y) + \ldots, \tag{4.4a}$$
$$v(x,y) = \epsilon^2 v_1(x,y) + \ldots, \tag{4.4b}$$
$$p(x,y) = \epsilon^2 p_2(x,y) + \ldots. \tag{4.4c}$$

Here $U_0(y) = f'_B(\bar{y})$ and is the velocity profile at $x^* = 0$ as given by the Blasius solution. The function $u_{\frac{1}{2}}(y)$ is a constant multiple of $g_{\frac{1}{2}}(\bar{y})$ as introduced in (3.14), and results from formally letting x^* tend to zero in the perturbation of order $\epsilon^{\frac{1}{2}}$ caused by the pressure variation of the same order. This term and the term $O(\epsilon \log \epsilon)$ are the only ones that differ from I (equation (3.1)). The presence of the latter is indicated by the form taken by the solution at the outer edge of the inner deck, and reference to it is made again in §5. Neither, however, makes a contribution to $u_1(x,y)$ since both are functions of y alone. The boundary conditions satisfied by u_1, v_1, p_2 upstream of the triple-deck region are obtained from the part of the perturbation Blasius solution of §3 that has a singular derivative as $x^* \to 0-$. The relevant matching is provided by (3.14) and we have

$$\frac{u_1}{(-x)^{\frac{1}{6}}} \to \alpha \frac{6^{\frac{1}{3}}(-\frac{1}{3})!}{\lambda^{\frac{1}{8}}} \frac{dU_0}{dy}, \quad (-x)^{\frac{5}{6}} v_1 \to \frac{\alpha(-\frac{1}{3})!}{6^{\frac{2}{3}} \lambda^{\frac{1}{8}}} U_0(y), \quad \frac{p_2}{(-x)^{\frac{1}{2}}} \to -\frac{\alpha}{\lambda^{\frac{1}{8}}}, \tag{4.5}$$

as $x \to -\infty$ with $y = O(1)$. On substituting (4.4) into the full Navier–Stokes equations and equating the coefficients of the leading powers of ϵ to zero we obtain

$$u_1(x,y) = A_1(x) \frac{dU_0}{dy}, \quad v_1(x,y) = -A'_1(x)\, U_0(y), \quad p_2(x,y) = p_2(x,0), \tag{4.6}$$

as in I (equations (3.6) and (3.8)), where $A_1(x)$ is a function of x to be determined. One equation connecting p_2 and A_1 follows from the upper deck in which $y^* = O(\epsilon^3 l)$ and in which the governing equations are essentially inviscid. To obtain this relation we introduce a new variable

$$Y = \epsilon\lambda^{\frac{5}{8}}y = \lambda^{\frac{5}{4}}y^*/l\epsilon^3, \tag{4.7}$$

and write

$$u = 1+\epsilon^2 U_2(x, Y)+\ldots, \quad v = \epsilon^2 V_2(x, Y)+\ldots, \quad p = \epsilon^2 P_2(x, Y)+\ldots, \tag{4.8}$$

where the dots denote higher powers of ϵ. Then it may easily be shown that P_2+iV_2 is a function of $x+iY$ only and that

$$P_2(x, 0) = p_2(x, 0), \quad V_2(x, 0) = -A_1'(x). \tag{4.9}$$

In I it was straightforward to express p_2 in terms of A_1' by means of a Hilbert integral but there is a slight complication here as $p_2(x, 0) \sim -\alpha(-x)^{\frac{1}{2}}\lambda^{-\frac{1}{8}}$ as $x \to -\infty$ and $A_1'(x) \sim \alpha x^{\frac{1}{2}}\lambda^{-\frac{1}{8}}$ as $x \to +\infty$ so that formally the Hilbert integrals do not converge. However the difficulty can be overcome by using Hadamard's notion of the finite part of the infinite integral and then we have

$$p_2(x, 0) = \frac{1}{\pi}\mathscr{F}\int_{-\infty}^{\infty}\frac{A_1'(x')}{x-x'}\,dx', \tag{4.10}$$

where $\mathscr{F}$ means that the finite part only is to be taken and that the integral is a Cauchy principal value. An alternative form is

$$p_2(x, 0) = \frac{-\alpha}{\lambda^{\frac{1}{8}}}(-x)^{\frac{1}{2}}H(-x)+\frac{1}{\pi}\int_{-\infty}^{0}\frac{A_1'(x')\,dx'}{x-x'}+\frac{1}{\pi}\int_{0}^{\infty}\frac{[A_1'(x')-\alpha\lambda^{-\frac{1}{8}}x'^{\frac{1}{2}}]}{x-x'}\,dx', \tag{4.11}$$

where H is Heaviside's step function.

Turning now to the lower deck of thickness $O(\epsilon^5 l)$ we write

$$z = \lambda^{\frac{1}{8}}y/\epsilon = \lambda^{\frac{3}{4}}y^*/l\epsilon^5 \tag{4.12a}$$

and
$$u = \epsilon\lambda^{\frac{1}{4}}\tilde{u}_1(x,z)+\ldots, \quad v = \epsilon^3\lambda^{\frac{1}{8}}\tilde{v}_1(x,z)+\ldots, \quad p = \epsilon^2\tilde{p}_2(x)+\ldots, \tag{4.12b}$$

where $\tilde{p}_2(x) = \lambda^{\frac{1}{8}}p_2(x, 0)$. Then $\tilde{u}_1$, $\tilde{v}_1$ satisfy

$$\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial x}+\tilde{v}_1\frac{\partial\tilde{u}_1}{\partial z} = -\frac{d\tilde{p}_2}{dx}+\frac{\partial^2\tilde{u}_1}{\partial z^2}, \quad \frac{\partial\tilde{u}_1}{\partial x}+\frac{\partial\tilde{v}_1}{\partial z} = 0, \tag{4.13}$$

with boundary conditions for $x < 0$,

$$\left.\begin{aligned} \tilde{u}_1 = 0 = \tilde{v}_1 \quad \text{if} \quad z = 0, \qquad \tilde{u}_1 - z \to \tilde{A}_1(x) \quad \text{as} \quad z \to \infty, \\ \tilde{u}_1 - z \to 0 \quad \text{as} \quad x \to -\infty, \end{aligned}\right\} \tag{4.14}$$

where $\tilde{A}_1(x) = \lambda^{\frac{1}{4}}A_1(x)$.

No boundary conditions have so far been given for (4.13) in $x > 0$ and before setting these out it is convenient to go through the analogous argument for the region $y^* < 0$. The only difference in the key equations (4.11), (4.13), (4.14) is that the sign of the term corresponding to $A_1(x)$ changes while the term corresponding to $p_2(x, 0)$ remains unaltered. Of course since we are dealing with an asymmetric problem the two boundary layers must be solved separately and no simple connecting relations can be expected. If we denote the value of $\tilde{p}_2(x)$ by $\tilde{p}_T(x)$ when $y^* > 0$ and by $\tilde{p}_B(x)$ when $y^* < 0$ with a corresponding notation for

$\tilde{A}_T(x)$ and $\tilde{A}_B(x)$ the fundamental problem of the triple deck for a lifting flat plate can be stated as follows.

Solve (4.13) with

$$\left.\begin{aligned}\tilde{p}_2(x) = \tilde{p}_T(x) &= \frac{1}{\pi}\mathscr{F}\int_{-\infty}^{\infty}\frac{\tilde{A}'_T(x')\,dx'}{x-x'} \quad \text{if} \quad z > 0,\\ \tilde{p}_2(x) = \tilde{p}_B(x) &= -\frac{1}{\pi}\mathscr{F}\int_{-\infty}^{\infty}\frac{\tilde{A}'_B(x')\,dx'}{x-x'} \quad \text{if} \quad z < 0,\end{aligned}\right\} \tag{4.15a}$$

subject to the following boundary conditions:

$$\tilde{u}_1 \to |z| \quad \text{as} \quad x \to -\infty; \tag{4.15b}$$

$$\tilde{u}_1 = 0 = \tilde{v}_1 \text{ at } z = 0, \quad x < 0, \quad \text{while } \partial\tilde{u}/\partial z \text{ is discontinuous}; \tag{4.15c}$$

$$\tilde{u}_1 - z \to \tilde{A}_T(x) \quad \text{as} \quad z \to +\infty, \quad \tilde{u}_1 + z \to -\tilde{A}_B(x) \quad \text{as} \quad z \to -\infty; \tag{4.15d}$$

$\tilde{u}_1$, $\tilde{v}_1$ are smooth for all z if $x > 0$;

$$\tilde{p}_T(x) = \tilde{p}_B(x) \quad \text{if} \quad x > 0. \tag{4.15e}$$

Finally $$\tilde{p}_T + \alpha(-x)^{\frac{1}{2}} \to 0, \quad \tilde{p}_B(x) - \alpha(-x)^{\frac{1}{2}} \to 0 \quad \text{as} \quad x \to -\infty. \tag{4.15f}$$

In the wake region a simple Galilean transformation can be made which, while not perhaps reducing the formidable numerical problem presented by (4.13), (4.15), makes it easier to see how to proceed and to understand the structure of the solution. In $x > 0$ we write

$$\tilde{z} = z - \theta(x), \tag{4.16}$$

where $\theta(x)$ is an arbitrary function of x, regard $\tilde{u}_1$ as a function of $x, \tilde{z}$, and replace $\tilde{v}_1$ by $\tilde{v}_1 + \theta'(x)\tilde{u}_1$. Then (4.13) is unaltered but the boundary conditions in $x > 0$ reduce to

$$\begin{aligned}\tilde{u}_1 - |\tilde{z}| \to {}& \tfrac{1}{2}[\tilde{A}_T(x) - \tilde{A}_B(x)]\\ &+ \{\tfrac{1}{2}[\tilde{A}_T(x) + \tilde{A}_B(x)] + \theta(x)\}\operatorname{sgn}\tilde{z} \quad \text{as} \quad |\tilde{z}| \to \infty.\end{aligned} \tag{4.17}$$

One possible choice for $\theta(x)$ is $-\frac{1}{2}(\tilde{A}_T + \tilde{A}_B)$ which simplifies (4.17) but, with an iterative method as outlined below, it is undesirable to move the origin. Hence we shall choose

$$\theta(0) = 0, \quad \theta' = -\tfrac{1}{2}(\tilde{A}'_T + \tilde{A}'_B) \quad (x > 0), \tag{4.18}$$

so that
$$\begin{aligned}\tilde{u}_1 - |\tilde{z}| \to {}& \tfrac{1}{2}[\tilde{A}_T(x) - \tilde{A}_B(x)]\\ &+ \tfrac{1}{2}\{\tilde{A}_T(0) + \tilde{A}_B(0)\}\operatorname{sgn}\tilde{z} \quad \text{as} \quad |\tilde{z}| \to \infty.\end{aligned} \tag{4.19}$$

The numerical integration might now proceed as follows. Guess $\tilde{A}_T$, $\tilde{A}_B$ in $x < 0$ and $\tilde{A}_T - \tilde{A}_B$ in $x > 0$. Using (4.15a) determine $\tilde{p}_T$, $\tilde{p}_B$ in $x < 0$ and $\tilde{p}_2$ in $x > 0$ together with the values of $\tilde{A}'_T + \tilde{A}'_B$ which make $\tilde{p}_T = \tilde{p}_B$ in $x > 0$. Then $\theta(x)$ is determined from (4.18). Now integrate (4.13) with these values of $\tilde{p}_2$ and θ to deduce new values of $\tilde{A}_T$, $\tilde{A}_B$ in $x < 0$ and of $\tilde{A}_T - \tilde{A}_B$ in $x > 0$. Hopefully this iterative procedure will converge to the required solution of the fundamental problem. Alternatively (4.15a) can be used in reverse, i.e. begin with $\tilde{p}_2$ and deduce $\tilde{A}'_T$, $\tilde{A}'_B$ from (4.15a). Then $\tilde{A}_T$, $\tilde{A}_B$ follow by integration, the additive constant being determined from the known properties of $\tilde{A}$ when x is large and negative. The last step in the cycle is to use (4.13) to compute $\tilde{p}_2$.

5. The structure of the lower deck

(*a*) $|x| \gg 1$, $x < 0$

The elucidation of the structure of the solution of (4.13) subject to (4.14), (4.15) cannot proceed straightforwardly, even on an intuitive basis, because the behaviour of the pressures $\tilde{p}_T$ and $\tilde{p}_B$ depends on the overall properties of $\tilde{A}_T$ and $\tilde{A}_B$. However we can expect that $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$, the anti-symmetric part of $\tilde{p}_2$, which is zero when $x > 0$ and is derived from a complex function of $x + iY$, has an asymptotic expansion for large negative x containing terms of the form $a_n(-x)^{\frac{1}{2}-n}$ with $n = 0, 1, 2, \ldots$. For $n \geqslant 1$ the coefficients a_n depend on the overall values of $\tilde{A}(x)$ while $a_0 = -\alpha$. Further, $\frac{1}{2}(\tilde{p}_T + \tilde{p}_B)$, the symmetric part of $\tilde{p}_2$, is, when x is large and negative, mainly forced by the wake growth at large positive values of x. This would imply, from I (equation (5.14)), that it has an asymptotic expansion which starts with $-1{\cdot}7840/3^{\frac{3}{2}}(-x)^{\frac{2}{3}}$. The dependence of the symmetric part of $\tilde{p}_2$ on overall properties of $\tilde{A}$ results in multipole solutions giving terms like $(-x)^{-n}$ for integral n. Otherwise the various terms arise from the properties of $\tilde{A}(x)$ when $|x|$ is large which depend in turn on the properties of $\tilde{p}_2$ and the eigensolutions of (4.13). Finally, logarithmic terms may arise through a confluence of forced terms and eigensolutions. On the basis of this general argument we therefore assume that

$$\tilde{p}_T(x) = -\alpha(-x)^{\frac{1}{2}} + \frac{\alpha a_1}{(-x)^{\frac{1}{2}}} - \frac{1.7840}{3^{\frac{3}{2}}(-x)^{\frac{2}{3}}} + O((-x)^{-\frac{5}{6}}), \tag{5.1}$$

when x is large and negative and verify *a posteriori* that it is a consistent assumption. The constant a_1 is related to the unknown circulation term B of (2.2) by

$$B = \epsilon^3 l \lambda^{-\frac{5}{4}} a_1, \tag{5.2}$$

and in § 6 an estimate of its value is made.

Following the argument of § 5 in I we now assume that the solution of (4.13) for the region $z > 0$ of the lower deck can be obtained in the form

$$\tilde{u}_1 = z + \alpha(-x)^{\frac{1}{6}} H_1'(\eta) + \alpha^2 H_2'(\eta) + \alpha^3(-x)^{-\frac{1}{6}} H_3'(\eta) + O((-x)^{-\frac{1}{3}}) \tag{5.3}$$

when x is large and negative. Here

$$\eta = z/3\,|2x|^{\frac{1}{3}}, \tag{5.4}$$

and the $H_n'(\eta)$ are functions of η satisfying the boundary conditions

$$H_n(0) = H_n'(0) = 0, \quad H_n''(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{5.5}$$

and the differential equations

$$H_n''' - 18\eta^2 H_n'' + 9(4-n)(\eta H_n' - H_n) = h_n(\eta). \tag{5.6}$$

Each $h_n(\eta)$ depends on the previous $H_m(\eta)$, $1 \leqslant m \leqslant n-1$, and

$$h_1(\eta) = 9\,.\,2^{-\frac{1}{3}}, \quad h_2(\eta) = 3\,.\,2^{-\frac{1}{3}}(3H_1 H_1'' - H_1'^2). \tag{5.7}$$

The second and third terms of (5.1) do not affect the expansion (5.3) until we reach $H_7(\eta)$, $H_8(\eta)$. The complementary functions of (5.6) are either exponentially large as $\eta \to \infty$, which is inadmissible, or linear, or, except for $n = 1, 2$, are such that their first derivative vanishes at infinity. If $n = 1$ one complementary function is $O(\eta^{\frac{1}{2}})$ as $\eta \to \infty$ and indeed we find that, on solving (5.6) analytically in this case,

$$H_1'(\eta) = -\frac{3^{\frac{1}{2}}(-\frac{2}{3})!}{(-\frac{1}{6})!}\eta^{\frac{1}{2}} + 6^{\frac{1}{3}}(-\frac{1}{3})! + O(\eta^{-\frac{1}{2}}) \tag{5.8}$$

as $\eta \to \infty$. Also

$$H_1''(0) = 3^{\frac{2}{3}}\left[\frac{(-\frac{2}{3})!}{(-\frac{1}{3})!}\right]^2, \tag{5.9}$$

and this gives a contribution to the skin friction that exactly matches with the corresponding term in (3.17). The first term of (5.8) matches with the second term of (4.4a) as forecast in the discussion of § 3. We note that this term is independent of x and so persists as $x \to +\infty$ and must appear in the expansion of $\tilde{u}_1$ about $x = \infty$. The second term of (5.8) gives a leading term in the asymptotic expansion of $\tilde{A}(x)$ so that

$$\tilde{A}_T(x) \approx \alpha 6^{\frac{1}{3}}(-\tfrac{1}{3})!\,(-x)^{\frac{1}{6}}. \tag{5.10}$$

The same term leads the asymptotic expansion of $\tilde{A}_B(x)$, and it follows from (4.15a) that (5.10) makes a contribution to $\tilde{p}_T(x)$ which is $O((-x)^{-\frac{5}{6}})$ and accounts for the last term of (5.1).

One complementary function of the equation for $H_2(\eta)$ is such that

$$H_2'(\eta) \sim \log \eta \quad \text{as} \quad \eta \to \infty,$$

and this presumably matches with the third term of (4.4a). This third term is $O(\log y)$ as $y \to 0$ and hence must be matched all the way along the lower deck even as $x \to +\infty$. The contribution to $\tilde{A}_T(x)$ arising from $H_2'(\eta)$ is a constant plus a term proportional to $\log |x|$, and gives a term in $\tilde{p}_T(x)$ which is proportional to $x^{-1}\log |x|$.

Thus, from conditions when x is large and negative, it would seem that the expansion (5.3) is the correct one: we shall confirm below that it is also consistent with the expansion as $x \to +\infty$. Through the kind offices of Dr N. Riley the first four equations of (5.7) were integrated numerically with the same basic program as was used to calculate the corresponding functions in Riley & Stewartson (1969) and it was found that

$$\left.\frac{\partial \tilde{u}_1}{\partial z}\right|_{z=0} = 1 - 2{\cdot}1539\sigma - 0{\cdot}8940\sigma^2 - 1{\cdot}2256\sigma^3 - 2{\cdot}2452\sigma^4 - \ldots, \tag{5.11}$$

where $\sigma = \alpha/(-x)^{\frac{1}{6}}$. In order to have a smooth solution it seems important to prevent separation occurring in the lower deck. It is clear from (5.1) that as x increases from $-\infty$ the pressure initially increases and so separation is a possibility. On the other hand the presence of the third term of (5.1) shows that the wake part of the lower deck provides a favourable pressure gradient which, although weak at large negative x, may well be enough to prevent separation if α is not too large. Certainly no separation occurs if $\alpha = 0$, and it is a reasonable hypothesis, in view of the existence of this term, to postulate the existence of an α_s such that if $\alpha < \alpha_s$ there is no separation and the triple-deck structure assumed here is correct, while if $\alpha > \alpha_s$ separation occurs and with it at least the partial collapse of the structure we have set up. We also postulate that α_s is associated with stall and define α_s as the trailing-edge stall angle. Clearly the determination of α_s is the most important end-point of the present theory but equally it presents a numerical problem that is beyond our capabilities at present. A rough estimate of its value can however be obtained as follows.

We compute the position of separation on the assumption that $\tilde{p}_T(x)$ is exactly equal to $-\alpha(-x)^{\frac{1}{2}}$. This may be done on the same lines as in Riley & Stewartson (1969) and we find, from (5.11), that if $x = x_s$, $\sigma = \sigma_s$ at this point then

$$0{\cdot}307 < \sigma_s < 0{\cdot}364 \tag{5.12}$$

and that the probable value of σ_s is near 0·326. We now set $a_1 = 0$ and determine the relative contribution of the third term of (5.1) to the pressure gradient at $x = x_s$. If

$$\alpha(-x_s)^{\frac{7}{8}} = 2 \tag{5.13}$$

this term is only about 20% of the first term and separation is unlikely to be inhibited. If

$$\alpha(-x_s)^{\frac{7}{8}} = 0{\cdot}45 \tag{5.14}$$

the pressure gradient has been reduced to zero at $x = x_s$ and separation is likely to have been inhibited. We infer that

$$0{\cdot}45 < \alpha_s^8/(0{\cdot}326)^7 < 2,$$

so that

$$0{\cdot}33 < \alpha_s < 0{\cdot}41. \tag{5.15}$$

With a Reynolds number of 10^6 the relation (4.2) in conjunction with (5.15) gives an angle of incidence of approximately 2° for the onset of separation. Since experimentally trailing-edge stall does not occur until the angle of incidence is much larger, between 5° and 15°, this predicted angle is much too small. The discrepancy may in part be explained by the fact that the observed flow is probably turbulent. In turbulent flow the displacement effect is greater than in laminar flow, the adverse pressure gradient is thereby decreased, and the boundary layer will remain attached at the trailing edge of the aerofoil through increased angles of incidence.

On the lower side of the plate the pressure variation due to the incidence is favourable so no separation takes place there for any α. The form of the expansions for $\tilde{u}_1$ and for $\tilde{p}_B$ are similar to (5.1) and (5.3) for the upper side of the plate, and the asymptotic structure of the skin friction may be obtained from (5.11) by changing the sign of σ.

(*b*) $|x| \ll 1$

Turning now to the immediate neighbourhood of the trailing edge of the plate we first note that the conventional boundary-layer equations, with main stream as given by (3.1) with $\alpha^* < 0$ and $O(1)$, have been integrated numerically by Ackerberg (private communication) who finds a complicated singularity at $x^* = 0$ with an infinite skin friction there. In our case α^* is small and the interaction with the main stream is likely to keep the boundary-layer properties finite if $\alpha < \alpha_s$. We may expect, however, that as $x \to 0-$ the values of $\partial\tilde{u}_1/\partial z$ as $z \to 0\pm$ are different. We denote them by $\lambda_T(\alpha)$ and $\lambda_B(\alpha)$, and for reasons similar to those given in I they are expected to be finite with $\lambda_B < 0 < \lambda_T$.

Further, in view of the previous history of the boundary layers on the top and bottom of the plate, they will satisfy, for $\alpha > 0$,

$$|\lambda_B(\alpha)| > |\lambda_B(0)| = \lambda_1 = \lambda_T(0) > \lambda_T(\alpha), \tag{5.16}$$

where λ_1 is defined in I (§6). As was the case there, the pressure and pressure gradient should be bounded as $x \to 0-$, but as $x \to 0+$ the pressure gradient is $O(x^{-\frac{1}{3}})$ which is necessary to prevent $\tilde{A}'(x)$ from being singular at $x = 0+$. The transition of the solution from $x = 0-$ to $x = 0+$ is achieved by a generalization of the Rott & Hakkinen (1965) wake solution. We suppose that the velocity profile at $x = 0-$ is

$$\tilde{u}_1(0-, z) = \hat{u}_1(z) \quad \text{with} \quad \hat{u}_1'(0+) = \lambda_T, \quad \hat{u}_1'(0-) = \lambda_B, \tag{5.17}$$

where $\hat{u}_1(z)$ is to be computed, and also

$$d\tilde{p}_2/dx \quad \text{is finite at } x = 0-, \text{ and} \quad d\tilde{p}_2/dx \approx C_0 x^{-\frac{1}{3}} \quad \text{as} \quad x \to 0+, \tag{5.18}$$

where C_0 is a constant to be found. Then, if, near $z = 0$,

$$\tilde{u}_1(x, z) = \tfrac{1}{3}(\tfrac{1}{4}x)^{\frac{1}{3}}\, G_0'(\eta) \quad (x > 0), \tag{5.19}$$

$G_0(\eta)$ satisfies

$$G_0''' + 2G_0 G_0'' - G_0'^2 = 27C_0 2^{\frac{4}{3}}, \tag{5.20}$$

with boundary conditions

$$\left.\begin{array}{lll} G_0'(\eta) - 18\lambda_T \eta \to 0 & \text{as} & \eta \to \infty, \\ G_0'(\eta) - 18\lambda_B \eta \to 0 & \text{as} & \eta \to -\infty. \end{array}\right\} \tag{5.21}$$

These conditions ensure that the velocity profile (5.19) matches with (5.17) as $x \to 0+$. This could of course be achieved if finite constants replaced the zeros on the right-hand sides of (5.21), but the additional restriction that these constants must be zero is necessary to ensure that $\tilde{A}_T'(x)$ and $\tilde{A}_B'(x)$ are bounded as $x \to 0+$. A discussion of this point is made in I (§6) for the special case $\lambda_T = -\lambda_B$, and the conclusions reached there are also applicable here.

Solutions of equation (5.20) with boundary conditions (5.21) have been kindly obtained for the authors by Mr P. G. Williams for a range of values of the positive parameter $-\lambda_T/\lambda_B$. The results are given in table 1, where η_0 is defined to be the value of η at which G_0 vanishes.

$-\lambda_T/\lambda_B$	1·0	0·8	0·6	0·4	0·2	0
$\lambda_T^{\frac{1}{3}}\eta_0$	0	0·017	0·040	0·070	0·109	0·164
$G_0'(\eta_0)/\lambda_T^{\frac{2}{3}}$	4·28	4·03	3·86	3·82	4·02	4·44
$C_0/\lambda_T^{\frac{4}{3}}$	0·409	0·351	0·287	0·213	0·124	0

TABLE 1

If $\lambda_T < 0$ the governing equation has no solutions since $G_0 < 0$ for large enough positive η and the method of solution completely breaks down in the neighbourhood of $x = 0$. Part of the reason is no doubt connected with the change in the direction of propagation of small disturbances, but at the present stage of development of the theory any attempts to overcome the difficulty are bound to be speculative especially in view of the singularity at separation, which has also to be dealt with, and so we shall not pursue the matter.

Returning to the case $\lambda_T > 0$ we see that the streamline from the trailing edge is given by $\eta = \eta_0$, so that it has a vertical tangent there, and in addition the streamwise component of velocity on it is proportional to $x^{\frac{1}{3}}$. Its subsequent behaviour probably needs a complete numerical integration for elucidation but we note that as in the symmetrical problem, even if $\hat{u}_1$ and $\tilde{p}_2$ are completely known, the form of $\tilde{u}_1$ downstream is not fully determinate and depends on an infinite set of arbitrary constants. The reason of course is that $\tilde{A}(x)$ near $x = 0$ is not entirely dependent on the local values of $\tilde{p}_2(x)$, and in addition to the arbitrary constants mentioned in I (§6) there will be others associated with $\theta(x)$ as introduced in (4.16).

(*c*) $|x| \gg 1,\ x > 0$

Finally, we consider the properties of the solution when $x \gg 1$. Here it seems that the Goldstein solution for the inner wake (Goldstein 1930) is appropriate together with the transformation (4.16). For large x we write

$$\tilde{u}_1 = \tfrac{1}{3}(\tfrac{1}{4}x)^{\frac{1}{3}}\, g_0'(\bar{\eta}) + \dots, \tag{5.22}$$

the dots denoting terms which are smaller when x is large, and

$$\bar{\eta} = \frac{z - \theta(x)}{3\,|2x|^{\frac{1}{3}}}. \tag{5.23}$$

Here g_0 satisfies the same differential equation (5.20) as G_0, except that $C_0 = 0$, together with boundary conditions

$$g_0(0) = g_0''(0) = 0, \quad g_0''(\infty) = 18, \tag{5.24}$$

and $\theta(x)$ is defined by (4.18). Physically this means that the lower deck terminates in a wake which is similar to that for a symmetrically disposed plate except that it is displaced a distance $\theta(x)$ upwards due to the upwash of the inviscid flow behind the inclined plate. From the relation between $\tilde{A}_1'(x)$ and $\tilde{p}_2(x)$ and the property $\tilde{p}_T = \tilde{p}_B$ where $x > 0$ we have

$$\theta(x) = \tfrac{2}{3}\alpha x^{\frac{3}{2}} + 2\alpha a_1 x^{\frac{1}{2}} + \dots, \tag{5.25}$$

when x is large, which is in accord with the above physical description of the flow. The properties of the Goldstein inner wake imply that

$$\tfrac{1}{2}[\tilde{A}_T(x) - \tilde{A}_B(x)] = 1{\cdot}416(\tfrac{1}{4}x)^{\frac{1}{3}} + \dots, \tag{5.26}$$

which gives a pressure decaying like $x^{-\frac{5}{3}}$ as $x \to \infty$ as in the symmetrical situation. In order to determine further terms in (5.25), (5.26) we set up an asymptotic series for $\tilde{u}_1$ in descending powers of x, and ultimately of $\log x$ also, whose coefficients are functions of $\bar{\eta}$, the leading term being given by (5.22). Were it not for the boundary conditions due to the asymmetry of the problem the structure of this series would be the same as that in I (§5) and so we shall concentrate on the asymmetrical features which are in fact dominant. Of these the most important arises from the term $u_{\frac{1}{2}}(y)$ in (4.4*a*) which behaves like $|y|^{\frac{1}{2}}\operatorname{sgn} y$ as $y \to 0$ and which matches with (5.3) when x is large and negative. Since it is independent of x it must match with the asymptotic series for $\tilde{u}_1$ when x is large and positive. This can be achieved by taking the first two terms of the series for $\tilde{u}_1$ as

$$\tilde{u}_1 = \tfrac{1}{3}(\tfrac{1}{4}x)^{\frac{1}{3}}\, g_0'(\bar{\eta}) + \tfrac{1}{3}\alpha(\tfrac{1}{4}x)^{\frac{1}{6}}\, g_1'(\bar{\eta}), \tag{5.27}$$

where g_1 satisfies

$$g_1''' + 2g_0 g_1'' - \tfrac{3}{2}(g_0' g_1' - g_0'' g_1) = 0, \tag{5.28}$$

together with the boundary conditions

$$g_1'(\eta) + 9\,.\,2^{\frac{1}{2}}\frac{(-\frac{2}{3})!}{(-\frac{1}{6})!}|\bar{\eta}|^{\frac{1}{2}} \to d_1 \operatorname{sgn}\bar{\eta} \quad \text{as} \quad |\bar{\eta}| \to \infty, \tag{5.29}$$

where d_1 is a constant which we now determine. The contribution to the asymptotic expansions of both $\tilde{A}_T(x)$ and $\tilde{A}_B(x)$ from the term in d_1 is

$$\tfrac{1}{3}\alpha(\tfrac{1}{4}x)^{\frac{1}{6}}\, d_1, \tag{5.30}$$

and we now obtain d_1 by noting that (5.30) in conjunction with (5.10) and the identical result for $\tilde{A}_B(x)$ gives a contribution to the pressure which must vanish for large positive x. Hence

$$d_1 = 2^{\frac{5}{3}}\, 3^{\frac{5}{6}}(-\tfrac{1}{3})!. \tag{5.31}$$

The solution of (5.28) may now be obtained uniquely and we find after numerical integration that $g_1(0) = 10{\cdot}0$, $g_1''(0) = 46{\cdot}0$. The consequent contribution to $\theta(x)$ is $O(x^{\frac{1}{6}})$ and is, as anticipated, smaller than both the terms in (5.25). Further terms in the expansion of $\tilde{u}_1$ may be found if necessary, but we shall not pursue the matter beyond noting that it will involve an infinite set of arbitrary constants.

6. An approximate solution for the antisymmetric part of the pressure

At the end of § 4 we outlined a possible procedure for the numerical solution of the fundamental problem presented by (4.13), (4.15), and in view of the considerable complexity of such a computation we feel it worthwhile to derive an approximate solution which would yield the antisymmetric part of the pressure and the symmetric part of the function $\tilde{A}_1(x)$. This is made possible by the fact that $\tilde{p}_T(x) = \tilde{p}_B(x)$ in the wake so that the antisymmetric part of the pressure, $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$, is zero for $x > 0$. Equations (4.13) become tractable if they are linearized about the shear flow with which $\tilde{u}_1(x, z)$ merges at the outer edge of the lower deck. The resulting equation should yield a solution exhibiting the main properties of the flow if it is regarded as valid for $x < 0$ only, since it is not expected that the linear shear is a good first approximation in the wake. The method of Wiener and Hopf then enables the functions $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$, $\frac{1}{2}(\tilde{A}_T + \tilde{A}_B)$ to be determined for all x, the former vanishing for $x > 0$, from a solution for the boundary layer over the plate which is independent of the boundary layer in the wake.

Denoting by $\tilde{u}_T(x, z)$, $\tilde{u}_B(x, z)$ the values of $\tilde{u}_1(x, z)$ for $z > 0$ and $z < 0$ respectively, we write

$$\tilde{u}_T(x, z) = Z + \tilde{w}_T(x, Z), \quad \tilde{u}_B(x, z) = Z + \tilde{w}_B(x, Z), \tag{6.1}$$

where $Z = |z|$, in the appropriate forms of (4.13), neglect the non-linear terms and subtract to obtain

$$\left.\begin{aligned} Z\frac{\partial}{\partial x}(\tilde{w}_T-\tilde{w}_B)+(\tilde{v}_T+\tilde{v}_B) &= -\frac{d}{dx}(\tilde{p}_T-\tilde{p}_B)+\frac{\partial^2}{\partial Z^2}(\tilde{w}_T-\tilde{w}_B),\\ \frac{\partial}{\partial x}(\tilde{w}_T-\tilde{w}_B)+\frac{\partial}{\partial Z}(\tilde{v}_T+\tilde{v}_B) &= 0. \end{aligned}\right\} \tag{6.2}$$

The fundamental equation, which is to be considered for $x < 0$ only, is then obtained from (6.2) as

$$Z\frac{\partial^2 w}{\partial x \partial Z} = \frac{\partial^3 w}{\partial Z^3} \quad (x < 0), \tag{6.3}$$

where $w = \frac{1}{2}(\tilde{w}_T - \tilde{w}_B)$, and is to be solved subject to the conditions

$$\frac{\partial^2 w}{\partial Z^2} = Q(x) \quad \text{on} \quad Z = 0; \quad w \to \tfrac{1}{2}(\tilde{A}_T + \tilde{A}_B) \quad \text{as} \quad Z \to \infty. \tag{6.4}$$

Here $Q(x) = \frac{1}{2}d(\tilde{p}_T - \tilde{p}_B)/dx$, and the boundary condition as $Z \to \infty$ follows from (4.15d).

If the Fourier transform of $w(x, Z)$ is denoted by $\bar{w}(\omega, Z)$ so that

$$\bar{w}(\omega, Z) = \int_{-\infty}^{\infty} w(x, Z)\, e^{-i\omega x}\, dx, \tag{6.5}$$

then, since $w(x, Z)$ satisfies (6.3) for $x < 0$, we have

$$\frac{\partial \bar{w}}{\partial Z} = \frac{\bar{Q}_+(\omega)\, e^{-\frac{1}{6}i\pi}}{\mathrm{Ai}'(0)\,(\omega - i\delta)^{\frac{1}{3}}}\mathrm{Ai}\{\exp[\tfrac{1}{6}i\pi]\,(\omega - i\delta)^{\frac{1}{3}} Z\} + M_-(\omega, Z). \tag{6.6}$$

The function $\bar{Q}_+(\omega)$ is the Fourier transform of $Q(x)$, and the suffix plus indicates that it is a regular function of the complex variable ω for $\mathrm{Re}\,\omega > 0$ since we require that $Q(x) \equiv 0$ for $x > 0$. The solution (6.6) satisfies the boundary condition on $Z = 0$ for $x < 0$, and contains the additional function $M_-(\omega, Z)$ regular for $\mathrm{Re}\,\omega < 0$ as the equation and boundary conditions satisfied by $w(x, Z)$ for $x > 0$ are unspecified. The parameter δ is introduced for convenience and the limiting process $\delta \to 0+$ will be made in conclusion. The branch of the cube root in (6.6) is to be chosen so that the argument of the Airy function has positive real part as $\mathrm{Re}\,\omega \to \pm\infty$.

A relationship between $\bar{Q}_+(\omega)$ and $\bar{C}(\omega)$, which is defined to be the Fourier transform of $\frac{1}{2}d^2(\tilde{A}_T + \tilde{A}_B)/dx^2$, is obtained from the upper deck. With an obvious extension of the notation of (4.7), (4.8) we have that in the upper deck

$$\frac{\partial}{\partial Y}(P_{2T} - P_{2B}) = -\frac{\partial}{\partial x}(V_{2T} + V_{2B}) \quad (Y \geqslant 0), \tag{6.7}$$

and that $P_{2T} - P_{2B}$ is harmonic in the variables x and Y. Thus if $\bar{Q}_2(\omega, Y)$ is the Fourier transform of $\frac{1}{2}\partial(P_{2T} - P_{2B})/\partial x$ we obtain, using (4.9),

$$\bar{Q}_2(\omega, Y) = \lambda^{-\frac{1}{4}}\bar{Q}_+(\omega)\, e^{-|\omega| Y}. \tag{6.8}$$

The factor $\lambda^{-\frac{1}{3}}$ is required since $\tilde{p}_2(x) = \lambda^{\frac{1}{3}} p_2(x, 0)$. If we now differentiate (6.7) with respect to x and let $Y \to 0$ we deduce that

$$\frac{\partial Q_2}{\partial Y} = \lambda^{-\frac{1}{3}} \frac{1}{2} \frac{d^3}{dx^3} (\tilde{A}_T + \tilde{A}_B). \tag{6.9}$$

Finally, combining (6.8), (6.9), we have

$$-|\omega| \bar{Q}_+(\omega) = i\omega \bar{C}(\omega), \tag{6.10}$$

where $C(\omega)$ is the transform of $\frac{1}{2} d^2(\tilde{A}_T + \tilde{A}_B)/dx^2$.

We are now in a position to apply the second of the boundary conditions (6.4). It becomes

$$-\omega^2 \bar{w}(x, Z) \to \bar{C}(\omega) \quad \text{as} \quad Z \to \infty, \tag{6.11}$$

so that, from (6.6),

$$-\frac{\bar{C}(\omega)}{\omega^2} = \int_0^\infty \frac{\partial \bar{w}}{\partial Z} dZ = \frac{\bar{Q}_+(\omega)\, e^{-\frac{1}{3}i\pi}}{3\,\mathrm{Ai}'(0)\,(\omega - i\delta)^{\frac{2}{3}}} + N_-(\omega), \tag{6.12}$$

where

$$N_-(\omega) = \int_0^\infty M_-(\omega, Z)\, dZ.$$

The function $\bar{C}(\omega)$ may now be eliminated between (6.10) and (6.12), and, if $|\omega|$ is replaced by $(\omega - i\delta)^{\frac{1}{2}} (\omega + i\delta)^{\frac{1}{2}}$, the result of the elimination is

$$\bar{Q}_+(\omega)\, K_+(\omega) = \gamma\, e^{\frac{1}{3}i\pi} (\omega - i\delta)^{\frac{2}{3}} N_-(\omega)\, K_-(\omega), \tag{6.13}$$

where

$$\frac{K_+(\omega)}{K_-(\omega)} = K(\omega) = 1 + \gamma\, e^{-\frac{1}{6}i\pi} \frac{(\omega + i\delta)^{\frac{1}{2}}}{(\omega - i\delta)^{\frac{11}{6}}} \tag{6.14}$$

and

$$0 < \gamma = -3\,\mathrm{Ai}'(0) = 3^{\frac{2}{3}}/(-\tfrac{2}{3})!. \tag{6.15}$$

Equation (6.13) has been written with the left-hand side regular for $\mathrm{Re}\,\omega > 0$ and the right-hand side regular for $\mathrm{Re}\,\omega < 0$ on the assumption that the factorization (6.14) has been made. We now make the additional assumption, which may be justified *a posteriori*, that the region of regularity of the left-hand side of (6.13) may be extended to $\mathrm{Re}\,\omega > -\delta$, and that of the right-hand side to $\mathrm{Re}\,\omega < \delta$. The two sides are now equal and regular on a dense set of points, so, by analytic continuation, together they define a function which is regular everywhere. Before proceeding further it is convenient to perform the factorization of $K(\omega)$.

The function $K(\omega)$ is regular and non-zero in the ω plane cut along the positive imaginary axis from $i\delta$ to $i\infty$, and along the negative imaginary axis from $-i\delta$ to $-i\infty$. The factorization is carried out in the usual way (see, for example, Noble 1958), and we obtain

$$\frac{K'_-(\omega)}{K_-(\omega)} = \frac{11}{6(\omega - i\delta)} + \frac{2\gamma}{3\pi i} \int_0^\infty \frac{\sigma^{\frac{1}{2}}\, d\sigma}{(\gamma^2 - 3^{\frac{1}{2}}\gamma\sigma^{\frac{4}{3}} + \sigma^{\frac{8}{3}})(\sigma + i\omega)}, \tag{6.16}$$

and

$$\frac{K'_+(\omega)}{K_+(\omega)} = \frac{1}{2(\omega + i\delta)} + \frac{4\gamma}{3\pi i} \int_0^\infty \frac{\sigma^{\frac{1}{3}}\, d\sigma}{(\gamma^2 + \sigma^{\frac{8}{3}})(\sigma - i\omega)}. \tag{6.17}$$

We shall require in particular the values of $K_-(\omega)$, $K_+(\omega)$ for $\omega = -it\gamma^{\frac{3}{4}}$, $\omega = is\gamma^{\frac{3}{4}}$ respectively where t, s are real and positive. They are

$$K_-(-it\gamma^{\frac{3}{4}}) = t^{\frac{11}{6}} \exp\left[-\frac{2}{3\pi} \int_0^\infty \frac{\sigma^{\frac{1}{3}} \log(\sigma + t)\, d\sigma}{1 - 3^{\frac{1}{2}}\sigma^{\frac{4}{3}} + \sigma^{\frac{8}{3}}}\right], \tag{6.18}$$

and
$$K_+(is\gamma^{\frac{3}{4}}) = -s^{\frac{1}{2}}\exp\left[\frac{4}{3\pi}\int_0^\infty \frac{\sigma^{\frac{1}{3}}\log(\sigma+s)\,d\sigma}{1+\sigma^{\frac{8}{3}}}\right], \tag{6.19}$$

where the arbitrary multiplicative constant is chosen so that

$$\frac{K_+(\omega)}{i\omega}\to\gamma^{-\frac{3}{4}},\quad \frac{K_-(\omega)}{i\omega}\to\gamma^{-\frac{3}{4}}\quad\text{as}\quad \operatorname{Re}\omega\to+\infty. \tag{6.20}$$

We are now in a position to return to (6.13). We set both sides equal to a constant D so that

$$Q(x) = \frac{D}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{K_+(\omega)}\,d\omega. \tag{6.21}$$

Since $K_+(\omega)$ as given by (6.17) is regular and non-zero for $\operatorname{Re}\omega \geqslant 0$ and is asymptotic to ω as $|\omega|\to\infty$, we see that $Q(x)\equiv 0$ for $x>0$ as required. Once $\bar{Q}_+(\omega)$ is known $\bar{C}(\omega)$ is given by (6.10) and $(\partial\bar{w}/\partial Z)|_{Z=0}$ by (6.6). With the use of (6.14) where necessary the three Fourier transforms are inverted to give

$$\left.\begin{aligned}\frac{1}{2}\frac{d}{dx}(\tilde{p}_T-\tilde{p}_B) &= -\frac{D\gamma^{\frac{3}{4}}}{\pi}\int_0^\infty\frac{\exp(\gamma^{\frac{3}{4}}xt)\exp[I(t)]}{t^{\frac{1}{2}}(1+t^{\frac{8}{3}})}\,dt \quad\text{if}\quad x<0\\ &= 0 \quad\text{if}\quad x>0\end{aligned}\right\} \tag{6.22}$$

$$\left.\begin{aligned}\frac{1}{2}\frac{d^2}{dx^2}(\tilde{A}_T+\tilde{A}_B) &= \frac{D\gamma^{\frac{3}{4}}}{\pi}\int_0^\infty\frac{\exp(\gamma^{\frac{3}{4}}xt)\,t^{\frac{5}{6}}\exp[I(t)]}{1+t^{\frac{8}{3}}}\,dt \quad\text{if}\quad x<0\\ &= \frac{D\gamma^{\frac{3}{4}}}{\pi}\int_0^\infty\frac{\exp(-\gamma^{\frac{3}{4}}xs)\exp[-J(s)]}{s^{\frac{1}{2}}}\,ds \quad\text{if}\quad x>0\end{aligned}\right\}, \tag{6.23}$$

$$\left.\frac{\partial w}{\partial Z}\right|_{Z=0} = \frac{D\gamma^{\frac{1}{2}}}{\pi}\frac{(-\frac{2}{3})!}{3^{\frac{1}{3}}(-\frac{1}{3})!}\int_0^\infty\frac{\exp(\gamma^{\frac{3}{4}}xt)\exp[I(t)]}{t^{\frac{5}{6}}(1+t^{\frac{8}{3}})}\,dt \quad\text{if}\quad x<0. \tag{6.24}$$

Here $I(t)$, $J(s)$ are the integrals appearing in (6.18), (6.19):

$$I(t) = \frac{2}{3\pi}\int_0^\infty\frac{\sigma^{\frac{1}{3}}\log(\sigma+t)}{1-3^{\frac{1}{2}}\sigma^{\frac{4}{3}}+\sigma^{\frac{8}{3}}}\,d\sigma, \tag{6.25}$$

$$J(s) = \frac{4}{3\pi}\int_0^\infty\frac{\sigma^{\frac{1}{3}}\log(\sigma+s)}{1+\sigma^{\frac{8}{3}}}\,d\sigma. \tag{6.26}$$

No expression for $(\partial w/\partial Z)|_{Z=0}$ for $x>0$ is given in (6.24) since the original equation (6.3) only determines w for $x<0$. The constant D will be determined by the requirement (4.15f) which gives

$$(-x)^{\frac{1}{2}}\,d(\tilde{p}_T-\tilde{p}_B)/dx\to\alpha\quad\text{as}\quad x\to-\infty. \tag{6.27}$$

In order to deduce the forms taken by (6.22)–(6.24) for large and small $|x|$, we require the asymptotic expansions of $I(t)$, $J(s)$ for small and large values of t, s. Since we find, from (6.25), (6.26), that

$$I(t)-I(t^{-1}) = \tfrac{5}{6}\log t,\quad J(s)-J(s^{-1}) = \tfrac{1}{2}\log s, \tag{6.28}$$

we need only consider small values of the variables. The results are

$$I(t) = 2^{\frac{1}{2}}t\cos\frac{\pi}{8}-\frac{t^{\frac{4}{3}}}{3^{\frac{1}{2}}}-\frac{t^2}{2^{\frac{3}{2}}}+\frac{t^{\frac{8}{3}}}{2}-\frac{2^{\frac{1}{2}}}{3}t^3\sin\frac{\pi}{8}+\frac{t^4}{3\pi}(\log t-\tfrac{1}{4})+O(t^5). \tag{6.29}$$

$$J(s) = 2^{\frac{1}{2}}s\cos\frac{\pi}{8}-\frac{2}{3^{\frac{1}{2}}}s^{\frac{4}{3}}+\frac{s^2}{2^{\frac{3}{2}}}-\frac{2^{\frac{1}{2}}}{3}s^3\sin\frac{\pi}{8}-\frac{s^4}{3\pi}(\log s-\tfrac{1}{4})+O(s^5). \tag{6.30}$$

We now determine the constant D by using (6.22), (6.27), (6.29) and obtain

$$D = -\frac{\alpha\pi^{\frac{1}{2}}}{2\gamma^{\frac{3}{8}}}. \tag{6.31}$$

The quantities $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$, $\frac{1}{2}(\tilde{A}_T + \tilde{A}_B)$ are obtained from (6.22), (6.23) by integration. The arbitrary constant in the integration of (6.22) is determined by the requirement that the pressure be continuous at $x = 0$. Of the four arbitrary

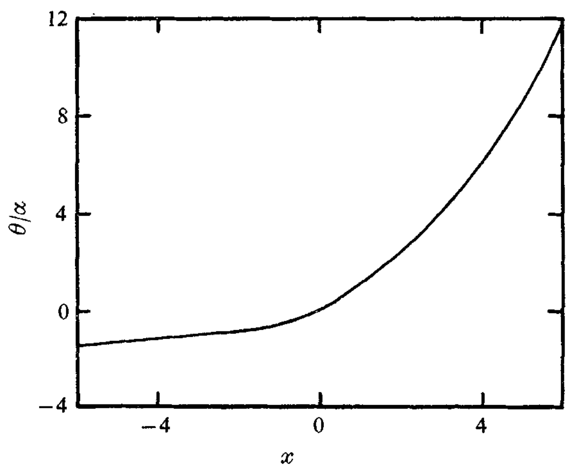

FIGURE 1. The function $\theta(x)/\alpha$.

constants that arise from (6.23), two are used to ensure that $\frac{1}{2}(\tilde{A}_T + \tilde{A}_B)$ and $\frac{1}{2}d(\tilde{A}_T + \tilde{A}_B)/dx$ are also continuous at the trailing edge, while with one of the remaining two we require that $\frac{1}{2}d(\tilde{A}_T + \tilde{A}_B)/dx \to 0$ as $x \to -\infty$ to comply with (5.10). The fourth constant is presumably determined by the constant in the asymptotic expansion of $H_2'(\eta)$ as $\eta \to \infty$ in (5.3), and is not at our disposal. However, since $\alpha^2 H_2'(\eta)$ represents a non-linear contribution to $\tilde{u}_1$ in (5.3) and the solution of this section embodies only the linear features of the fundamental problem of the trailing edge, we shall not match these two constants. The function plotted in figure 1 is $\theta(x)/\alpha$ with the definition of (4.18) extended to $x < 0$. For $x > 0$, $\theta(x)$ represents the deviation from the centre-line of the streamline that comes off the trailing edge of the plate. At $x = 0$, $\theta(x) = 0$ and $\theta'(x)$ is bounded, though $\theta''(x)$ is logarithmically infinite.

The resulting expressions for $(\tilde{p}_T - \tilde{p}_B)/2\alpha$, $(2\alpha)^{-1}\{(\partial\tilde{u}_1/\partial z)_T + (\partial\tilde{u}_1/\partial z)_B\}_{z=0}$ are illustrated in figure 2. As for $\theta(x)$, expansions were found for small and large values of $|x|$, and the integrals were evaluated numerically for intermediate values of x. For large $|x|$ the expansions follow easily from (6.29), (6.30) and we note the first few terms here in order to compare the results with the predictions of § 5.

If x is large and negative the appropriate forms are

$$\tfrac{1}{2}(\tilde{p}_T - \tilde{p}_B) = -\alpha\left\{(-x)^{\frac{1}{2}} - \frac{(-x)^{-\frac{1}{2}}}{2^{\frac{1}{2}}\gamma^{\frac{3}{4}}}\cos\frac{\pi}{8} + \frac{(-\frac{1}{3})!}{6^{\frac{2}{3}}3^{\frac{1}{2}}}(-x)^{-\frac{5}{6}} + O((-x)^{-\frac{3}{2}})\right\}, \tag{6.32}$$

$$\tfrac{1}{2}(\tilde{A}_T + \tilde{A}_B) = \alpha\{6^{\frac{1}{3}}(-\tfrac{1}{3})!\,(-x)^{\frac{1}{6}} + O(1)\}, \tag{6.33}$$

and

$$\frac{1}{2}\left\{\left(\frac{\partial \tilde{u}_1}{\partial z}\right)_T+\left(\frac{\partial \tilde{u}_1}{\partial z}\right)_B\right\}_{z=0} = -\frac{\alpha}{6^{\frac{1}{3}}}\left[\frac{(-\frac{2}{3})!}{(-\frac{1}{3})!}\right]^2\left\{(-x)^{-\frac{1}{6}}+\frac{(-x)^{-\frac{7}{6}}}{3\,.\,2^{\frac{1}{2}}\gamma^{\frac{3}{4}}}\cos\frac{\pi}{8}+O((-x)^{-\frac{3}{2}})\right\}, \tag{6.34}$$

while for $x > 0$

$$\tfrac{1}{2}(\tilde{A}_T+\tilde{A}_B) = -\alpha\left\{\tfrac{2}{3}x^{\frac{3}{2}}+\frac{2^{\frac{1}{2}}x^{\frac{1}{2}}}{\gamma^{\frac{3}{4}}}\cos\frac{\pi}{8}-\frac{2^{\frac{4}{3}}}{3^{\frac{1}{6}}}(-\tfrac{1}{3})!\,x^{\frac{1}{6}}+O(1)\right\}. \tag{6.35}$$

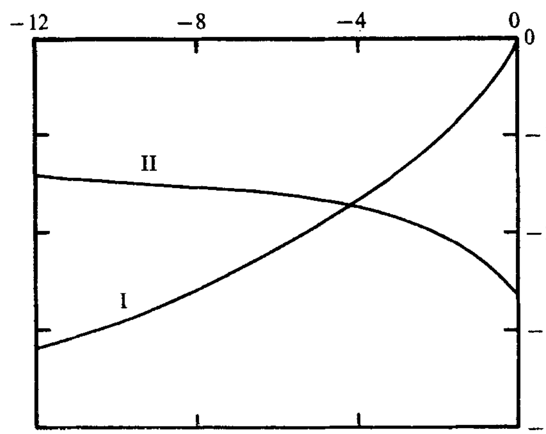

FIGURE 2. I, The anti-symmetric pressure $(\tilde{p}_T-\tilde{p}_B)/2\alpha$; II, The symmetric skin friction

$$\frac{1}{2\alpha}\left\{\left(\frac{\partial \tilde{u}_1}{\partial z}\right)_T+\left(\frac{\partial \tilde{u}_1}{\partial z}\right)_B\right\}_{z=0}.$$

This linearized solution of the problem of the trailing edge is consistent with, and gives confidence in, the structure set out in the preceding sections. Comparison of (6.32) with (5.1) gives an estimate of the previously unknown constant a_1 as

$$a_1 = 2^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}\cos\tfrac{1}{8}\pi = 0{\cdot}7898, \tag{6.36}$$

and then the two expressions are seen to agree except for the additional term $O((-x)^{-\frac{2}{3}})$ in (5.1). However, as this arises from the symmetric part of the pressure it is automatically excluded from (6.32). Similarly, we may compare (6.33) with (5.10), and (6.34) with (5.3) and (5.8). Finally, we note the agreement between (6.34) and (5.25) and (5.31).

The constant a_1 of which we have an estimate in (6.36) is related to the circulation term B of (2.2) by (5.2), and the assumption of §3 that $B = O(\epsilon^3 l)$ is seen to be justified. Thus there is a stagnation point of the outer inviscid flow on the upper side of the plate at a distance from the trailing edge given by

$$-x^*/l = \epsilon^7\alpha^2 a_1^2\lambda^{-\frac{1}{4}}. \tag{6.37}$$

7. Supersonic trailing edges

It is of interest to compare and contrast the results for incompressible flow with those for supersonic flow. We suppose that the flat plate is fixed in a compressible fluid which has Mach number $M_\infty > 1$ at an infinite distance upstream. Thus at the leading edge of the plate an expansion fan is formed on the upper side and a shock on the lower side. According to inviscid theory the slip velocity and pressure on the plate are given by

$$\left.\begin{aligned} U_1(x^*) &= U_\infty + U_\infty \alpha^* \operatorname{sgn} y^*/(M_\infty^2 - 1)^{\frac{1}{2}}, \\ p^* &= p_\infty - U_\infty^2 \rho_\infty \alpha^* \operatorname{sgn} y^*/(M_\infty^2 - 1)^{\frac{1}{2}}. \end{aligned}\right\} \tag{7.1}$$

At the trailing edge a triple deck similar to that in incompressible flow is set up, the main difference arising from the inviscid-flow properties in the upper deck where the governing equation is the linear wave equation instead of the potential equation. The structure in fact is the same as that proposed by Stewartson & Williams (1969) for the closely related problem of self-induced separation and which is based on ideas introduced by Lighthill (1953). We write for $y^* > 0$ in the lower deck

$$\left.\begin{aligned} \frac{p^* - p_\infty}{\rho_\infty U_\infty^2} &= e^2 \frac{C^{\frac{1}{4}} \lambda^{\frac{1}{2}}}{(M_\infty^2 - 1)^{\frac{1}{4}}} \tilde{p}_T(x), \\ \frac{x^*}{l} &= e^3 \frac{C^{\frac{3}{8}} \lambda^{-\frac{5}{4}}}{(M_\infty^2 - 1)^{\frac{3}{8}}} \left(\frac{T_w}{T_\infty}\right)^{\frac{3}{2}} x, \\ \frac{y^*}{l} &= e^5 \frac{C^{\frac{5}{8}} \lambda^{-\frac{3}{4}}}{(M_\infty^2 - 1)^{\frac{1}{8}}} \left(\frac{T_w}{T_\infty}\right)^{\frac{3}{2}} z, \\ \frac{u^*(x^*, y^*)}{U_\infty} &= e \frac{C^{\frac{1}{8}} \lambda^{\frac{1}{4}}}{(M_\infty^2 - 1)^{\frac{1}{8}}} \left(\frac{T_w}{T_\infty}\right)^{\frac{1}{2}} \tilde{u}_1(x, z), \\ \frac{v^*(x^*, y^*)}{U_\infty} &= e^3 C^{\frac{3}{8}} \lambda^{\frac{3}{4}} (M_\infty^2 - 1)^{\frac{1}{8}} \left(\frac{T_w}{T_\infty}\right)^{\frac{1}{2}} \tilde{v}_1(x, z), \end{aligned}\right\} \tag{7.2}$$

as in (5.11)–(5.15) of Stewartson & Williams (1969), where T_w is the wall temperature, T_∞ the temperature at infinity, and C is Chapman's constant which occurs in the linear viscosity law

$$\mu/\mu_\infty = CT/T_\infty, \quad C = \mu_w T_\infty/\mu_\infty T_w \tag{7.3}$$

discussed in Stewartson (1964, p. 35), for example, and μ is the coefficient of viscosity. With these assumptions the fundamental boundary-layer equation for the region $z > 0$ is the same as (4.13). The boundary conditions in $x < 0$ are the same as in (4.14) the only difference being the relation between $\tilde{p}_T(x)$ and $\tilde{A}_T(x)$. Instead of (4.15a) we now have

$$\tilde{p}_T(x) = -\frac{d\tilde{A}_T}{dx} - \alpha^* \frac{C^{-\frac{1}{4}} \lambda^{-\frac{1}{2}}}{e^2 (M_\infty^2 - 1)^{\frac{1}{4}}}, \tag{7.4}$$

the difference being due to the change in structure of the upper deck and to the fact that on leaving the triple deck in the upstream direction $\tilde{A}_T$ tends to zero but $\tilde{p}_T$ is given by (7.1). Similar results hold for the lower deck with some sign changes analogous to (4.15) [e.g. $\tilde{p}_T(x)$ is replaced by $-\tilde{p}_B(x)$ in (7.4)].

The form of the two lower decks in supersonic flow is quite different from that in incompressible flow because now there is no possibility of the solution at a particular station of x directly affecting what happens farther upstream. Instead, on both the top and bottom of the plate, there occurs a self-induced flow due to the non-uniqueness of the governing equation (4.13) when subject to the condition (7.4). Some properties of this solution were discussed in Stewartson & Williams (1969) where it was shown that for a rising pressure the only disposable parameter is x_0 which fixes the position of separation. In the present problem, if $\alpha^* = 0$, we need the other solution in which the pressure falls as x increases so that the skin friction increases. At the trailing edge the skin friction is greater than that given by Blasius and is the same on the upper and lower sides of the plate. Just downstream of the trailing edge the Rott & Hakkinen (1965) similarity solution holds in the neighbourhood of the line $z = 0$ with pressure gradient proportional to $x^{-\frac{1}{3}}$. Consequently $\tilde{A}_T(x)$ and $\tilde{p}_T(x)$ vary linearly as $x \to 0+$ but their second derivatives are singular. The formal solution is then continued by forward numerical integration and proceeds until, as $x \to \infty$, the Goldstein wake is approached with pressure gradient proportional to $x^{-\frac{5}{3}}$. Presumably the disposable parameter in this solution that fixes the pressure at $x = 0-$ is determined by the condition that $\tilde{p}_T(x) \to 0$ as $x \to \infty$.

When $\alpha^* > 0$ it is clear from (7.4) that the crucial parameter is α_c where

$$\alpha^* = \epsilon^2 C^{\frac{1}{4}} \lambda^{\frac{1}{2}} (M_\infty^2 - 1)^{\frac{1}{4}} \alpha_c. \tag{7.5}$$

If $\alpha_c \ll 1$ the effect of the inclination of the plate to the main stream may be neglected in comparison with that due to the trailing edge. If $\alpha_c = O(1)$, then there are two disposable constants x_T and x_B fixing the self-induced solutions of (4.13) on top and on the bottom of the plate. On the lower side the pressure will fall, and the appropriate solution is the same as for the trailing edge of a symmetrically disposed plate though $x_B \neq x_0$. On the top the pressure depends on the value of α_c and may fall or rise. The difference $x_T - x_B$ is determined by the condition that $\tilde{p}_T(0) = \tilde{p}_B(0)$. Thereafter the same procedure is used as when $\alpha_c = 0$ and x_T is determined by the condition $\tilde{p}_T(\infty) = 0$. If the pressure rises there is a possibility of separation on the upper side and once this occurs the development of the flow is not clear. Although the separation is not accompanied by a singularity as explained in Stewartson & Williams (1969, § 9), the equation of Rott & Hakkinen (1965) does not appear to have a solution when reversed flow has occurred on one side of the plate. A further point to note is that it has not yet been determined what ultimately happens to the self-induced solution in which the pressure decreases with increasing x. It appears possible that $\tilde{p}(x) \to -\infty$ as $x \to \infty$ but further numerical work is required before firm conclusions may be drawn. All we can confirm at the moment is that separation occurs when α^*, the angle of incidence of the wing, is such that

$$\alpha^* \sim \left[\frac{C(M_\infty^2 - 1)}{R}\right]^{\frac{1}{4}} \tag{7.6}$$

in contrast to the result $\alpha^* = O(R^{-\frac{1}{16}})$ in the incompressible case.

Finally we observe that, if the compressible flow is subsonic rather than supersonic, the appropriate scaling in the triple deck is the same as (7.2) except that (M_∞^2-1) is replaced by $(1-M_\infty^2)$. In addition, in (3.1), α^* is replaced by $\alpha^*/(1-M_\infty^2)^{\frac{1}{2}}$. The condition for separation then becomes

$$\alpha^* \sim \epsilon^{\frac{1}{2}} C^{\frac{1}{16}} \left(\frac{T_w}{T_\infty}\right)^{-\frac{3}{4}} (1-M_\infty^2)^{\frac{7}{16}}. \tag{7.7}$$

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